Nonabelian Toda equations
associated with classical Lie groups

A. V. Razumov, M. V. Saveliev, A. B. Zuevsky

Institute for High Energy Physics, 142284 Protvino, Moscow Region, Russia

Abstract

The grading operators for all nonequivalent $\mathbb{Z}$-gradations of classical Lie algebras are represented in the explicit block matrix form. The explicit form of the corresponding nonabelian Toda equations is given.

1   Introduction

Toda equations arise in many problems of modern theoretical and mathematical physics. There is a lot of papers devoted to classical and quantum behaviour of abelian Toda equations. From the other hand, nonabelian Toda equations have not yet received a due attention. From our point of view, this is mainly caused by the fact that despite of their formal exact integrability till recent time there were no nontrivial examples of nonabelian Toda equations for which one can write the general solution in a more or less explicit form. Moreover, even the form of nonabelian Toda equations was known only for a few partial cases. In our recent paper [1] we described some class of nonabelian Toda equations called there maximally nonabelian. These equations have a very simple structure and their general solution can be explicitly written. Shortly after that we realised that the approach used in [1] allows to describe the explicit form of all nonabelian Toda equations associated with classical Lie groups. This is done in the present paper.

2 $\mathbb{Z}$-gradations and Toda equations

2.1   Toda equations

From the point of view of the group-algebraic approach [2, 3] Toda equations are specified by a choice of a real or complex Lie group whose Lie algebra is endowed with a $\mathbb{Z}$-gradation. Recall that a Lie algebra $\mathfrak{g}$ is said to be $\mathbb{Z}$-graded, or endowed with a $\mathbb{Z}$-gradation, if there is given a representation of $\mathfrak{g}$ as a direct sum

$$\mathfrak{g} = \bigoplus_{m \in \mathbb{Z}} \mathfrak{g}_m,$$

1

where $[\mathfrak{g}_m, \mathfrak{g}_n] \subset \mathfrak{g}_{m+n}$ for all $m, n \in \mathbb{Z}$.

Let G be a real or complex Lie group, and $\mathfrak{g}$ be its Lie algebra. For a given $\mathbb{Z}$-gradation of $\mathfrak{g}$ the subspace $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}$. The subspaces

$$\mathfrak{g}_{<0} = \bigoplus_{m<0} \mathfrak{g}_m, \qquad \mathfrak{g}_{>0} = \bigoplus_{m>0} \mathfrak{g}_m$$

are also subalgebras of $\mathfrak{g}$. Denote by G_0, $G_{<0}$ and $G_{>0}$ the connected Lie subgroups of G corresponding to the subalgebras $\mathfrak{g}_0$, and $\mathfrak{g}_{<0}$ and $\mathfrak{g}_{>0}$ respectively.

Let M be either the real manifold $\mathbb{R}^2$ or the complex manifold $\mathbb{C}$. For $M = \mathbb{R}^2$ we denote the standard coordinates by z^- and z^+. In the case of $M = \mathbb{C}$ we use the notation z^- for the standard complex coordinate and z^+ for the complex conjugate of z^-. Denote the partial derivatives over z^- and z^+ by ∂_- and ∂_+ respectively. Consider a Lie group G whose Lie algebra $\mathfrak{g}$ is endowed with a $\mathbb{Z}$-gradation. Let l be a positive integer, such that the grading subspaces $\mathfrak{g}_m$ for $-l < m < 0$ and $0 < m < l$ are trivial, and c_- and c_+ be some fixed mappings from M to $\mathfrak{g}_{-l}$ and $\mathfrak{g}_{+l}$, respectively, satisfying the relations

$$\partial_+ c_- = 0, \qquad \partial_- c_+ = 0.$$

Restrict ourselves to the case when G is a matrix Lie group. In this case the Toda equations are matrix partial differential equations of the form

$$\partial_+(\gamma^{-1}\partial_-\gamma) = [c_-, \gamma^{-1}c_+\gamma], \tag{2.1}$$

where γ is a mapping from M to G_0. If the Lie group G_0 is abelian we say that we deal with abelian Toda equations, otherwise we call them nonabelian Toda equations.

There is a constructive procedure of obtaining the general solution to Toda equations [3, 4]. It is based on the use of the Gauss decomposition related to the $\mathbb{Z}$-gradation under consideration. Here the Gauss decomposition is the representation of an element of the Lie group G as a product of elements of the subgroups $G_{<0}$, $G_{>0}$ and G_0 taken in an appropriate order. Another approach is based on the theory of representations of Lie groups [2, 3].

2.2 $\mathbb{Z}$-gradations of complex semisimple Lie algebras

Let q be an element of a Lie algebra $\mathfrak{g}$ such that the linear operator $\operatorname{ad} q$ is semisimple and has integer eigenvalues. Defining

$$\mathfrak{g}_m = \{x \in \mathfrak{g} \mid [q, x] = mx\}$$

we get a $\mathbb{Z}$-gradation of $\mathfrak{g}$. This gradation is said to be generated by the grading operator q. If $\mathfrak{g}$ is a finite dimensional complex semisimple Lie algebra, then any $\mathbb{Z}$-gradations of $\mathfrak{g}$ is generated by a grading operator. Here up to the action of the

group of the automorphisms of $\mathfrak{g}$ all $\mathbb{Z}$-gradations of $\mathfrak{g}$ can be obtained with the help of the following procedure.

Let Δ be the set of roots of a complex semisimple Lie algebra $\mathfrak{g}$ with respect to a Cartan subalgebra $\mathfrak{h}$, and $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ be a base of Δ. Assign to the vertices of the Dynkin diagram of $\mathfrak{g}$ nonnegative integer labels q_i, $i = 1, \ldots, r$, and define

$$q = \sum_{i,j=1}^{r} h_i(k^{-1})_{ij} q_j \tag{2.2}$$

where h_i are the corresponding Cartan generators and $k = (k_{ij})$ is the Cartan matrix of $\mathfrak{g}$. It is clear that q is a grading operator of some $\mathbb{Z}$-gradation of $\mathfrak{g}$. Here the subspace $\mathfrak{g}_m$ for $m \neq 0$ is the direct sum of the root subspaces $\mathfrak{g}^\alpha$ corresponding to the roots $\alpha = \sum_{i=1}^{r} n_i \alpha_i$ with $\sum_{i=1}^{r} n_i q_i = m$. The subspace $\mathfrak{g}_0$, besides the root subspaces corresponding to the roots $\alpha = \sum_{i=1}^{r} n_i \alpha_i$ with $\sum_{i=1}^{r} n_i q_i = 0$, includes the Cartan subalgebra $\mathfrak{h}$.

If all labels q_i are different from zero, then the subgroup $\mathfrak{g}_0$ coincides with the Cartan subalgebra of $\mathfrak{h}$. In this case the subgroup G_0 is abelian. In all other cases the subgroup G_0 is nonabelian. The maximally nonabelian Toda equations [1] arise in the case when only one of the labels q_i is different from zero.

2.3   Conformal invariance

Let again G be a real or complex Lie group, $\mathfrak{g}$ be its Lie algebra, and M be either the real manifold $\mathbb{R}^2$ or the complex manifold $\mathbb{C}$. Since M is simply connected a connection on the trivial principal G-bundle $M \times G$ can be identified with a $\mathfrak{g}$-valued 1-form ω on M. Here the connection is flat if and only if

$$\mathrm{d}\omega + \omega \wedge \omega = 0. \tag{2.3}$$

We call this relation the zero curvature condition. It can be shown that the Toda equations coincide with the zero curvature condition for the connection

$$\omega = \mathrm{d}z^-(\gamma^{-1}\partial_-\gamma + c_-) + \mathrm{d}z^+\gamma^{-1}c_+\gamma. \tag{2.4}$$

Let $\xi_\pm$ be some mappings from M to G_0, satisfying the condition

$$\partial_+\xi_- = 0, \qquad \partial_-\xi_+ = 0,$$

and γ be a solution of the Toda equations (2.1). It is easy to get convinced that the mapping

$$\gamma' = \xi_+^{-1}\gamma\xi_- \tag{2.5}$$

satisfies the Toda equations (2.1) with the mappings $c_\pm$ replaced by the mappings

$$c'_\pm = \xi_\pm^{-1}c_\pm\xi_\pm.$$

3

In this sense, the Toda equations determined by the mappings $c_\pm$ and $c'_\pm$ which are connected by the above relation, are equivalent. If the mappings $\xi_\pm$ are such that

$$\xi_\pm^{-1} c_\pm \xi_\pm = c_\pm,$$

then transformation (2.5) is a symmetry transformation for the Toda equations.

Let us show that if the $\mathbb{Z}$-gradation under consideration is generated by a grading operator and c_- and c_+ are constant mappings, then the corresponding Toda equations are conformally invariant. Let $F : M \to M$ be a conformal transformation. It means that for the functions $F^- = z^- \circ F$ and $F^+ = z^+ \circ F$ one has

$$\partial_+ F^- = 0, \qquad \partial_- F^+ = 0.$$

For the connection ω, given by (2.4), we get

$$F^* \omega = \mathrm{d}z^- [(\gamma \circ F)^{-1} \partial_- (\gamma \circ F) + \partial_- F^- c_-] + \mathrm{d}z^+ (\gamma \circ F)^{-1} \partial_+ F^+ c_+ (\gamma \circ F).$$

If the connection ω satisfies the zero curvature condition (2.3), then the connection $F^* \omega$ also satisfies this condition. So if the mapping γ satisfies the Toda equations, then the mapping $\gamma \circ F$ satisfies the equations

$$\partial_+ [(\gamma \circ F)^{-1} \partial_- (\gamma \circ F)] = \partial_- F^- \partial_+ F^+ [c_-, (\gamma \circ F)^{-1} c_+ (\gamma \circ F)].$$

It is always possible to compensate the factor $\partial_- F^- \partial_+ F^+$ in the right hand side of the above equation with the help of transformation (2.5). Indeed, defining

$$\xi_- = \exp\left(-q\, l^{-1} \ln \partial_- F^-\right), \qquad \xi_+ = \exp\left(q\, l^{-1} \ln \partial_+ F^+\right),$$

one obtains

$$\xi_-^{-1} c_- \xi_- = (\partial_- F^-)^{-1} c_-, \qquad \xi_+^{-1} c_+ \xi_+ = (\partial_+ F^+)^{-1} c_+.$$

Therefore, the mapping

$$\gamma' = \exp\left(-q\, l^{-1} \ln \partial_+ F^+\right) (\gamma \circ F) \exp\left(-q\, l^{-1} \ln \partial_- F^-\right). \tag{2.6}$$

satisfies the initial Toda equations. Thus, transformation (2.6) is a symmetry transformation for the Toda equations. Such transformations define an action of the group of conformal transformations on the space of solutions of the Toda equations under consideration.

3   Complex general linear group

We begin the consideration of nonabelian Toda systems associated with classical Lie groups with the case of the Lie group $\mathrm{SL}(r+1, \mathbb{C})$. Actually it is convenient to

consider the Lie group $\mathrm{GL}(r+1,\mathbb{C})$ whose Lie algebra $\mathfrak{gl}(r+1,\mathbb{C})$ is endowed with $\mathbb{Z}$-gradations induced by $\mathbb{Z}$-gradations of the Lie algebra $\mathfrak{sl}(r+1,\mathbb{C})$.

The Lie algebra $\mathfrak{sl}(r+1,\mathbb{C})$ is of type A_r. The Cartan matrix is

$$
k = \begin{pmatrix}
2 & -1 & 0 & \cdots & 0 & 0 & 0 \\
-1 & 2 & -1 & \cdots & 0 & 0 & 0 \\
0 & -1 & 2 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 2 & -1 & 0 \\
0 & 0 & 0 & \cdots & -1 & 2 & -1 \\
0 & 0 & 0 & \cdots & 0 & -1 & 2
\end{pmatrix}.
$$

For the inverse matrix one obtains the expression

$$
k^{-1} = \frac{1}{r+1} \begin{pmatrix}
r & r-1 & r-2 & \cdots & 3 & 2 & 1 \\
r-1 & 2(r-1) & 2(r-2) & \cdots & 6 & 4 & 2 \\
r-2 & 2(r-2) & 3(r-2) & \cdots & 9 & 6 & 3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
3 & 6 & 9 & \cdots & 3(r-2) & 2(r-2) & r-2 \\
2 & 4 & 6 & \cdots & 2(r-2) & 2(r-1) & r-1 \\
1 & 2 & 3 & \cdots & r-2 & r-1 & r
\end{pmatrix}.
$$

Let d be a fixed integer such that $1 \leq d \leq r$. Consider the $\mathbb{Z}$-gradation of $\mathfrak{sl}(r+1,\mathbb{C})$ arising when we choose the labels of the corresponding Dynkin diagram equal to zero except the label q_d which is chosen equal to 1. From relation (2.2) it follows that the corresponding grading operator, which we denote by $\overset{(d)}{q}$, has the form

$$
\overset{(d)}{q} = \frac{1}{r+1} \left[(r+1-d) \sum_{i=1}^{d-1} i h_i + d \sum_{i=d}^{r} (r+1-i) h_i \right].
$$

It is convenient to take as a Cartan subalgebra of $\mathfrak{sl}(r+1,\mathbb{C})$ the subalgebra consisting of diagonal $(r+1)\times(r+1)$ matrices with zero trace. Here the standard choice of the Cartan generators is

$$
h_i = e_{i,i} - e_{i+1,i+1},
$$

where the matrices $e_{i,j}$ are defined by

$$
(e_{i,j})_{kl} = \delta_{ik}\delta_{jl}. \tag{3.1}
$$

With such a choice of Cartan generators we obtain

$$
\overset{(d)}{q} = \frac{1}{r+1} \left[(r+1-d) \sum_{i=1}^{d} e_{i,i} - d \sum_{i=d+1}^{r+1} e_{i,i} \right].
$$

Thus, the grading operator has the following block matrix form:

$$\overset{(d)}{q} = \frac{1}{r+1} \begin{pmatrix} k_2 I_{k_1} & 0 \\ 0 & -k_1 I_{k_2} \end{pmatrix},$$

where $k_1 = d$ and $k_2 = r + 1 - d$, so that $k_1 + k_2 = r + 1$. Here and henceforth I_k denotes the unit $k \times k$ matrix.

The grading operator corresponding to the general $\mathbb{Z}$-gradation of $\mathfrak{sl}(r+1, \mathbb{C})$ is a linear combination of the operators $\overset{(d)}{q}$ with nonnegative integer coefficients. The explicit matrix form of the grading operators is depicted as follows. A general set of grading labels q_i can be represented as

$$(\underbrace{0, \ldots, 0}_{k_1-1}, m_1, \underbrace{0, \ldots, 0}_{k_2-1}, m_2, 0, \ldots, 0, m_{p-1}, \underbrace{0, \ldots, 0}_{k_p-1}),$$

where $k_1, \ldots, k_p$ and $m_1, \ldots, m_{p-1}$ are positive integers. It is convenient to consider an arbitrary matrix x of $\mathfrak{sl}(r+1, \mathbb{C})$ as a $p \times p$ block matrix (x_{ab}), where x_{ab} is a $k_a \times k_b$ matrix. The grading operator corresponding to the above set of labels has the following block matrix form:

$$q = \begin{pmatrix} \rho_1 I_{k_1} & 0 & \cdots & 0 & 0 \\ 0 & \rho_2 I_{k_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \rho_{p-1} I_{k_{p-1}} & 0 \\ 0 & 0 & \cdots & 0 & \rho_p I_{k_p} \end{pmatrix}, \tag{3.2}$$

where

$$\rho_a = \frac{1}{r+1} \left(-\sum_{b=1}^{a-1} m_b \sum_{c=1}^{b} k_c + \sum_{b=a}^{p-1} m_b \sum_{c=b+1}^{p} k_c \right).$$

We will use grading operator (3.2) to define a $\mathbb{Z}$-gradation of the Lie algebra $\mathfrak{gl}(r+1, \mathbb{C})$. It is easy to describe the arising grading subspaces of $\mathfrak{gl}(r+1, \mathbb{C})$ and the relevant subgroups of $\mathrm{GL}(r+1, \mathbb{C})$. For fixed $a \neq b$, the block matrices x having only the block x_{ab} different from zero belong to the grading subspace $\mathfrak{g}_m$ with

$$m = \sum_{c=a}^{b-1} m_c, \quad a < b, \qquad m = \sum_{c=b}^{a-1} m_c, \quad a > b.$$

The block diagonal matrices form the subalgebra $\mathfrak{g}_0$. The subalgebras $\mathfrak{g}_{<0}$ and $\mathfrak{g}_{>0}$ are formed by all block strictly lower and upper triangular matrices respectively. It is not difficult to describe the corresponding subgroups. The subgroup G_0 consists of all block diagonal nondegenerate matrices, and the subgroups $G_{<0}$ and $G_{>0}$ consist, respectively, of all block lower and upper triangular matrices with unit matrices on the diagonal. Note that the subgroup G_0 is isomorphic to the Lie group $\mathrm{GL}(k_1, \mathbb{C}) \times \cdots \times \mathrm{GL}(k_p, \mathbb{C})$.

6

Proceed now to the consideration of the corresponding Toda equations. Assume that all integers m_a are equal to one. In this case one has

$$\rho_a = \frac{1}{r+1} \sum_{b=1}^{p} bk_b - a.$$

The elements c_- and c_+ should belong to the subspaces $\mathfrak{g}_{-1}$ and $\mathfrak{g}_{+1}$ respectively. The general form of such elements is

$$c_- = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ C_{-1} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & C_{-(p-1)} & 0 \end{pmatrix}, \quad c_+ = \begin{pmatrix} 0 & C_{+1} & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & C_{+(p-1)} \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad (3.3)$$

where for each $a = 1, \ldots, p-1$ the mapping C_{-a} takes values in the space of $k_{a+1} \times k_a$ complex matrices, and the mapping C_{+a} takes values in the space of $k_a \times k_{a+1}$ complex matrices. Besides, these mappings should satisfy the relations

$$\partial_+ C_{-a} = 0, \qquad \partial_- C_{+a} = 0.$$

Parametrise the mapping γ as

$$\gamma = \begin{pmatrix} \beta_1 & 0 & \cdots & 0 & 0 \\ 0 & \beta_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & \beta_{p-1} & 0 \\ 0 & 0 & \cdots & 0 & \beta_p \end{pmatrix}, \quad (3.4)$$

where the mappings β_a take values in the groups $\mathrm{GL}(m_a, \mathbb{C})$. In this parametrisation Toda equations (2.1) take the form

$$\partial_+(\beta_1^{-1} \partial_- \beta_1) = -\beta_1^{-1} C_{+1} \beta_2 C_{-1}, \tag{3.5}$$
$$\partial_+(\beta_a^{-1} \partial_- \beta_a) = -\beta_a^{-1} C_{+a} \beta_{a+1} C_{-a} + C_{-(a-1)} \beta_{a-1}^{-1} C_{+(a-1)} \beta_a, \quad 1 < a < p, \tag{3.6}$$
$$\partial_+(\beta_p^{-1} \partial_- \beta_p) = C_{-(p-1)} \beta_{p-1}^{-1} C_{+(p-1)} \beta_p. \tag{3.7}$$

The consideration of more general $\mathbb{Z}$-gradations gives nothing new. Indeed, the $\mathbb{Z}$-gradations with all integers m_a equal to 1 exhaust all possible subgroups G_0. Furthermore, the mappings $c_\pm$ corresponding to a general $\mathbb{Z}$-gradations should take values in subalgebras $\mathfrak{g}_{\pm l}$, where l is less or equal to the minimal value of the positive integers m_a. It is clear that the blocks $(c_\pm)_{ab}$ are nonzero only if $|a - b| = 1$. Therefore, the general form of the mappings $c_\pm$ is again given by (3.3), where the mappings $C_{\pm a}$ corresponding to the grading indexes greater than l should be zero mappings.

4 Complex orthogonal group

It is convenient for our purposes to define the complex orthogonal group $\mathrm{O}(n,\mathbb{C})$ as the Lie subgroup of the Lie group $\mathrm{GL}(n,\mathbb{C})$ formed by matrices $a \in \mathrm{GL}(n,\mathbb{C})$ satisfying the condition

$$\tilde{I}_n a^t \tilde{I}_n = a^{-1}, \tag{4.1}$$

where $\tilde{I}_n$ is the antidiagonal unit $n\times n$ matrix, and a^t is the transpose of a. The corresponding Lie algebra $\mathfrak{o}(n,\mathbb{C})$ is the subalgebra of $\mathfrak{gl}(n,\mathbb{C})$ which consists of the matrices x satisfying the condition

$$\tilde{I}_n x^t \tilde{I}_n = -x. \tag{4.2}$$

For a $k_1 \times k_2$ matrix a we will denote by a^T the matrix defined by the relation

$$a^T = \tilde{I}_{k_2} a^t \tilde{I}_{k_1}.$$

Using this notation, we can rewrite conditions (4.1) and (4.2) as $a^T = a^{-1}$ and $x^T = -x$. The Lie algebra $\mathfrak{o}(n,\mathbb{C})$ is simple. For $n = 2r+1$ it is of type B_r, while for $n = 2r$ it is of type D_r. Discuss these two cases separately.

Consider the $\mathbb{Z}$-gradation of $\mathfrak{o}(2r+1,\mathbb{C})$ arising when we choose $q_d = 1$ for some fixed d such that $1 \le d \le r$, and put all other labels of the Dynkin diagram be equal to zero. The Cartan matrix for the Lie algebra $\mathfrak{o}(2r+1,\mathbb{C})$ is given by

$$k = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -2 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix},$$

and for its inverse one has the expression

$$k^{-1} = \frac{1}{2} \begin{pmatrix} 2 & 2 & 2 & \cdots & 2 & 2 & 2 \\ 2 & 4 & 4 & \cdots & 4 & 4 & 4 \\ 2 & 4 & 6 & \cdots & 6 & 6 & 6 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 2 & 4 & 6 & \cdots & 2(r-2) & 2(r-2) & 2(r-2) \\ 2 & 4 & 6 & \cdots & 2(r-2) & 2(r-1) & 2(r-1) \\ 1 & 2 & 3 & \cdots & r-2 & r-1 & r \end{pmatrix}.$$

Using relation (2.2), one gets

$$\overset{(d)}{q} = \sum_{i=1}^{r-1} i h_i + \frac{1}{2} r h_r, \quad d = r,$$

$$\overset{(d)}{q} = \sum_{i=1}^{r-1} ih_i + \frac{1}{2}(r-1)h_r, \quad d = r-1,$$

$$\overset{(d)}{q} = \sum_{i=1}^{d} ih_i + d \sum_{i=d+1}^{r-1} h_i + \frac{1}{2}dh_r, \quad 1 \le d < r-1.$$

It is convenient to choose the following Cartan generators of $\mathfrak{o}(2r+1, \mathbb{C})$:

$$h_i = e_{i,i} - e_{i+1,i+1} + e_{2r+1-i,2r+1-i} - e_{2r+2-i,2r+2-i}, \quad 1 \le i < r,$$
$$h_r = 2(e_{r,r} - e_{r+2,r+2}),$$

where the matrices $e_{i,j}$ are defined by (3.1). Using these expressions one obtains

$$\overset{(d)}{q} = \sum_{i=1}^{d} e_{i,i} - \sum_{i=1}^{d} e_{2r+2-i,2r+2-i}.$$

Denoting $k_1 = d$ and $k_2 = 2(r-d) + 1$, we write q in block matrix form,

$$\overset{(d)}{q} = \begin{pmatrix} I_{k_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -I_{k_1} \end{pmatrix}, \tag{4.3}$$

where zero on the diagonal stands for the $k_2 \times k_2$ block of zeros.

The Cartan matrix for the Lie algebra $\mathfrak{o}(2r, \mathbb{C})$ has the form

$$k = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 0 & 2 \end{pmatrix},$$

and its inverse is

$$k^{-1} = \frac{1}{4} \begin{pmatrix} 4 & 4 & 4 & \cdots & 4 & 2 & 2 \\ 4 & 8 & 8 & \cdots & 8 & 4 & 4 \\ 4 & 8 & 12 & \cdots & 12 & 6 & 6 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 4 & 8 & 12 & \cdots & 4(r-2) & 2(r-2) & 2(r-2) \\ 2 & 4 & 6 & \cdots & 2(r-2) & r & r-2 \\ 2 & 4 & 6 & \cdots & 2(r-2) & r-2 & r \end{pmatrix}.$$

In this case one obtains

$$\overset{(d)}{q} = \frac{1}{2}\sum_{i=1}^{r-2} ih_i + \frac{1}{4}(r-2)h_{r-1} + \frac{1}{4}rh_r, \quad d = r,$$

$$\overset{(d)}{q} = \frac{1}{2}\sum_{i=1}^{r-2} ih_i + \frac{1}{4}rh_{r-1} + \frac{1}{4}(r-2)h_r, \quad d = r-1,$$

$$\overset{(d)}{q} = \sum_{i=1}^{d} ih_i + d\sum_{i=d+1}^{r-2} h_i + \frac{1}{2}d(h_{r-1} + h_r), \quad 1 \le d < r-1.$$

Choose as the Cartan generators of $\mathfrak{o}(2r, \mathbb{C})$ the elements

$$h_i = e_{i,i} - e_{i+1,i+1} + e_{2r-i,2r-i} - e_{2r+1-i,2r+1-i}, \quad 1 \le i < r,$$
$$h_r = e_{r-1,r-1} + e_{r,r} - e_{r+1,r+1} - e_{r+2,r+2}.$$

Then it is easy to see that

$$\overset{(d)}{q} = \frac{1}{2}\sum_{i=1}^{r} e_{i,i} - \frac{1}{2}\sum_{i=1}^{r} e_{2r+1-i,2r+1-i}, \quad d = r,$$

$$\overset{(d)}{q} = \frac{1}{2}\sum_{i=1}^{r-1} e_{i,i} - \frac{1}{2}e_{r,r} + \frac{1}{2}e_{r+1,r+1} - \frac{1}{2}\sum_{i=1}^{r-1} e_{2r+1-i,2r+1-i}, \quad d = r-1,$$

$$\overset{(d)}{q} = \sum_{i=1}^{d} e_{i,i} - \sum_{i=1}^{d} e_{2r+1-i,2r+1-i}, \quad 1 \le d < r-1.$$

Note that the grading operators corresponding to the cases $d = r$ and $d = r-1$ are connected by the automorphism σ of $\mathfrak{o}(2r, \mathbb{C})$ defined by the relation $\sigma(x) = axa^{-1}$, where

$$a = \begin{pmatrix} 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

There is the corresponding automorphism of the Lie group $O(2r, \mathbb{C})$, which is defined by the same formula. Thus, the cases $d = r$ and $d = r-1$ leads actually to the same $\mathbb{Z}$-gradation.

For the case $d = r$ the grading operator has the following block form

$$\overset{(r)}{q} = \frac{1}{2}\begin{pmatrix} I_k & 0 \\ 0 & -I_k \end{pmatrix}, \tag{4.4}$$

where we denoted $k = r$. In the case $1 \le d < r - 2$ denoting $k_1 = d$ and $k_2 = 2(r-d)$ one sees that the grading operator q has form (4.3).

The grading operator of a general $\mathbb{Z}$-gradation of the Lie algebra $\mathfrak{o}(n, \mathbb{C})$ is again a linear combination of the grading operators $\overset{(d)}{q}$ with non-negative integer coefficients. Using the explicit form of the operators $\overset{(d)}{q}$ and taking into account the existence of the automorphism of $\mathfrak{o}(2r, \mathbb{C})$ described above, we come to the following explicit description of the $\mathbb{Z}$-gradations of $\mathfrak{o}(n, \mathbb{C})$.

A $\mathbb{Z}$-gradation of $\mathfrak{o}(n, \mathbb{C})$ is determined first by a fixation of block matrix representation of the elements of $\mathfrak{o}(n, \mathbb{C})$. Here any element x is seen as a $p \times p$ block matrix (x_{ab}), where $p \le n$ and x_{ab} is a $k_a \times k_b$ matrix. Now the positive integers k_a are not arbitrary. They are restricted by the relation

$$k_a = k_{p-a+1}.$$

To get a concrete $\mathbb{Z}$-gradation, one also have to fix a set of positive integers m_a, $a = 1, \ldots, p - 1$, subjected to the constraint

$$m_a = m_{p-a}.$$

The corresponding grading operator has the form (3.2) with

$$\rho_a = \frac{1}{2} \left(-\sum_{b=1}^{a-1} m_b + \sum_{b=a}^{p-1} m_b \right).$$

The structure of the subalgebras $\mathfrak{g}_0$, $\mathfrak{g}_{<0}$, $\mathfrak{g}_{>0}$ and the corresponding subgroups is the same as in the case of general linear group with the exception that we should use only those block matrices which belong to $\mathrm{SO}(n, \mathbb{C})$. It is clear that the subgroup G_0 for an odd $p = 2s + 1$ is isomorphic to the Lie group $\mathrm{GL}(k_1, \mathbb{C}) \times \cdots \times \mathrm{GL}(k_s, \mathbb{C}) \times \mathrm{SO}(k_{s+1}, \mathbb{C})$ while for an even $p = 2s$ it is isomorphic to the Lie group $\mathrm{GL}(k_1, \mathbb{C}) \times \cdots \times \mathrm{GL}(k_s, \mathbb{C})$. Note that the latter is possible only if n is even.

Consider now the corresponding Toda equations. As for the case of the general linear group it suffices to consider only the $\mathbb{Z}$-gradations for which all integers m_a are equal to 1. In this case one has

$$\rho_a = \frac{p+1}{2} - a.$$

The general form of the mappings $c_\pm$ is given by (3.3) where

$$C_{\pm a}^T = -C_{\pm(p-a)}. \tag{4.5}$$

We will use the parametrisation of the mapping γ given by (3.4), where

$$\beta_a^T = \beta_{p-a+1}^{-1}. \tag{4.6}$$

So for the case $p = 2s + 1$ we have $s + 1$ independent mappings β_a and in the case $p = 2s$ there are s independent mappings.

The Toda equations has form (3.5)–(3.7), where the mappings $C_{\pm a}$ and β_a satisfy relations (4.5) and (4.6). In the case $p = 2s + 1$ for the independent mappings $\beta_1, \ldots, \beta_{s+1}$ one can write

$$\partial_+(\beta_1^{-1}\partial_-\beta_1) = -\beta_1^{-1}C_{+1}\beta_2 C_{-1},$$
$$\partial_+(\beta_a^{-1}\partial_-\beta_a) = -\beta_a^{-1}C_{+a}\beta_{a+1}C_{-a} + C_{-(a-1)}\beta_{a-1}^{-1}C_{+(a-1)}\beta_a, \quad 1 < a \leq s,$$
$$\partial_+(\beta_{s+1}^{-1}\partial_-\beta_{s+1}) = -\beta_{s+1}^T C_{+s}^T \beta_s^{-1T} C_{-s}^T + C_{-s}\beta_s^{-1}C_{+s}\beta_{s+1}.$$

Note that in this case $\beta_{s+1}^T = \beta_{s+1}^{-1}$. In the case $p = 2s$ the independent equations are

$$\partial_+(\beta_1^{-1}\partial_-\beta_1) = -\beta_1^{-1}C_{+1}\beta_2 C_{-1}, \tag{4.7}$$
$$\partial_+(\beta_a^{-1}\partial_-\beta_a) = -\beta_a^{-1}C_{+a}\beta_{a+1}C_{-a} + C_{-(a-1)}\beta_{a-1}^{-1}C_{+(a-1)}\beta_a, \quad 1 < a < s, \tag{4.8}$$
$$\partial_+(\beta_s^{-1}\partial_-\beta_s) = -\beta_s^{-1}C_{+s}\beta_s^{-1T}C_{-s} + C_{-(s-1)}\beta_{s-1}^{-1}C_{+(s-1)}\beta_s, \tag{4.9}$$

where $C_{-s}^T = -C_{-s}$ and $C_{+s}^T = -C_{+s}$.

5 Complex symplectic group

We define the complex symplectic group $\mathrm{Sp}(2r, \mathbb{C})$ as the Lie subgroup of the Lie group $\mathrm{GL}(2r, \mathbb{C})$ which consists of the matrices $a \in \mathrm{GL}(2r, \mathbb{C})$ satisfying the condition

$$\tilde{J}_r a^t \tilde{J}_r = -a^{-1},$$

where $\tilde{J}_r$ is the matrix given by

$$\tilde{J}_r = \begin{pmatrix} 0 & \tilde{I}_r \\ -\tilde{I}_r & 0 \end{pmatrix}.$$

The corresponding Lie algebra $\mathfrak{sp}(r, \mathbb{C})$ is defined as the subalgebra of the Lie algebra $\mathfrak{sl}(2r, \mathbb{C})$ formed by the matrices x which satisfy the condition

$$\tilde{J}_r x^t \tilde{J}_r = x.$$

The Lie algebra $\mathfrak{sp}(r, \mathbb{C})$ is simple, and it is of type C_r. Therefore, the Cartan matrix of $\mathfrak{sp}(r, \mathbb{C})$ is the transpose of the Cartan matrix of $\mathfrak{o}(2r, \mathbb{C})$, and the same is true for the inverse of the Cartan matrix of $\mathfrak{sp}(r, \mathbb{C})$. Thus, the explicit form of the Cartan matrix is

$$k = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -2 & 2 \end{pmatrix},$$

and for its inverse one has

$$k^{-1} = \frac{1}{2} \begin{pmatrix} 2 & 2 & 2 & \cdots & 2 & 2 & 1 \\ 2 & 4 & 4 & \cdots & 4 & 4 & 2 \\ 2 & 4 & 6 & \cdots & 6 & 6 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 2 & 4 & 6 & \cdots & 2(r-2) & 2(r-2) & r-2 \\ 2 & 4 & 6 & \cdots & 2(r-2) & 2(r-1) & r-1 \\ 2 & 4 & 6 & \cdots & 2(r-2) & 2(r-1) & r \end{pmatrix}.$$

For any fixed integer d such that $1 \le d \le r$, consider the $\mathbb{Z}$-gradation of $\mathfrak{sp}(r, \mathbb{C})$ arising when we choose all the labels of the corresponding Dynkin diagram equal to zero, except the label q_d, which we choose be equal to 1.

Using relation (2.2), we obtain the following expressions for the grading operator,

$$\overset{(d)}{q} = \frac{1}{2} \sum_{i=1}^{r} i h_i, \quad d = r, \qquad \overset{(d)}{q} = \sum_{i=1}^{d} i h_i + d \sum_{i=d+1}^{r} h_i, \quad 1 \le d < r.$$

Using the following choice of the Cartan generators,

$$h_i = e_{i,i} - e_{i+1,i+1} + e_{2r-i,2r-i} - e_{2r+1-i,2r+1-i}, \qquad 1 \le i < d,$$
$$h_r = e_{r,r} - e_{r+1,r+1},$$

one sees that the grading operator for the case $d = r$ has form (4.4) with $k = r$, and for the case $1 \le d < r$ it has form (4.3) with $k_1 = d$ and $k_2 = 2(r - d)$. So we have the same grading operators and, therefore, the same structure of grading subspaces as we had in the case of the Lie algebra $\mathfrak{so}(2r, \mathbb{C})$. In the case of odd $p = 2s + 1$ the subgroup G_0 is isomorphic to the Lie group $\mathrm{GL}(k_1, \mathbb{C}) \times \cdots \times \mathrm{GL}(k_s, \mathbb{C}) \times \mathrm{Sp}(k_{s+1}, \mathbb{C})$. Note that here k_{s+1} is even. In the case $p = 2s$ the subgroup G_0 is isomorphic to $\mathrm{GL}(k_1, \mathbb{C}) \times \cdots \times \mathrm{GL}(k_s, \mathbb{C})$.

Without any loss of generality we assume that all integers m_a characterising the $\mathbb{Z}$-gradation are equal to 1. The general form of the mappings $c_\pm$ is given by (3.3), where in the case $p = 2s + 1$ one has

$$C_{-a}^T = -C_{-(p-a)}, \qquad C_{+a}^T = -C_{+(p-a)}, \qquad a \ne s,$$
$$\tilde{I}_{k_s} C_{-s}^t \tilde{J}_{k_{s+1}/2} = -C_{-(s+1)}, \qquad \tilde{J}_{k_{s+1}/2} C_{+s}^t \tilde{I}_{k_s} = C_{+(s+1)}.$$

In the case $p = 2s$ the mappings $C_{\pm a}$ should satisfy the relations

$$C_{-a}^T = -C_{-(p-a)}, \qquad C_{+a}^T = -C_{+(p-a)}, \qquad a \ne s,$$
$$C_{-s}^T = C_{-s}, \qquad C_{+s}^T = C_{+s}.$$

To write the Toda equations in an explicit form we use again the parametrisation (3.4), where in the case $p = 2s + 1$ one has

$$\beta_a^T = \beta_{p-a+1}^{-1}, \qquad a \ne s + 1,$$
$$\tilde{J}_{k_{s+1}/2} \beta_{s+1}^t \tilde{J}_{k_{s+1}/2} = -\beta_{s+1}^{-1},$$

13

whereas in the case $p = 2s$

$$\beta_a^T = \beta_{p-a+1}^{-1}$$

for any $a = 1, \ldots, 2s$. The independent Toda equations in the case $p = 2s + 1$ are

$$\partial_+(\beta_1^{-1}\partial_-\beta_1) = -\beta_1^{-1}C_{+1}\beta_2 C_{-1},$$
$$\partial_+(\beta_a^{-1}\partial_-\beta_a) = -\beta_a^{-1}C_{+a}\beta_{a+1}C_{-a} + C_{-(a-1)}\beta_{a-1}^{-1}C_{+(a-1)}\beta_a, \quad 1 < a \leq s,$$
$$\partial_+(\beta_{s+1}^{-1}\partial_-\beta_{s+1}) = \beta_{s+1}^{-1}\tilde{J}_{k_{s+1}/2}C_{+s}^t\beta_s^{-1t}C_{-s}^t\tilde{J}_{k_{s+1}/2} + C_{-s}\beta_s^{-1}C_{+s}\beta_{s+1}.$$

In the case $p = 2s$ one has equations (4.7)–(4.9), where $C_{-s}^T = C_{-s}$ and $C_{+s}^T = C_{+s}$.

6   Concluding remarks

To construct the general solution for the equations described in the present paper one can apply the method based on the Gauss decomposition. For some partial cases this is done in our paper [1]. The method based on the representation theory was applied to this problem by A. N. Leznov [5, 6]. One can also use the methods considered by A. N. Leznov and E. A. Yusbashjan [7] and by P. Etingof, I. Gelfand and V. Retakh [8, 9] which lead to some very simple forms of the solution but, unfortunately, cannot be applied in general situation.

It is worth to note that all nonabelian Toda equations associated with the Lie groups $SO(n, \mathbb{C})$ and $Sp(n = 2m, \mathbb{C})$ can be obtained by reduction of appropriate equations associated with the Lie group $GL(n, \mathbb{C})$. Actually this fact can be proved without using concrete matrix realisation of the Lie groups and Lie algebras under consideration.[1]

The results obtained above can be generalised to the case of higher grading Toda equations [10, 11] and multidimensional Toda-type equations [4].

From the point of view of physical applications it is interesting to investigate possible reductions to real Lie groups. Some results in this direction valid for $\mathbb{Z}$-gradations generated by the Cartan generator of some $SL(2, \mathbb{C})$-subgroup of G are obtained by J. M. Evans and J. O. Madsen [12].

We believe that nonabelian Toda equations are quite relevant for a number of problems of theoretical and mathematical physics, and in a near future their role for the description of nonlinear phenomena in many areas will be not less than that of the abelian Toda equations.

Acknowledgements

---

[1]We are thankful to A. N. Leznov for the discussion of this point.

References

[1] A. V. Razumov and M. V. Saveliev, Maximally non-abelian Toda systems, *Nucl. Phys.* **B 494** (1997) 657–686, hep-th/9612081.

[2] A. N. Leznov and M. V. Saveliev, *Group-theoretical methods for integration of nonlinear dynamical systems* (Birkhauser, Basel, 1992).

[3] A. V. Razumov and M. V. Saveliev, *Lie algebras, geometry, and Toda-type systems* (Cambridge University Press, Cambridge, 1997).

[4] A. V. Razumov and M. V. Saveliev, Multi-dimensional Toda-type systems, *Theor. Math. Phys.* **112** (1997) 999–1022; hep-th/9609031.

[5] A. N. Leznov, The exactly integrable systems connected with semisimple algebras of the second rank A_2, B_2, C_2, G_2, math-ph/9809012.

[6] A. N. Leznov, Graded Lie algebras, representation theory, integrable mappings and systems: nonabelian case, math-ph/9810006.

[7] A. N. Leznov and E. A. Yusbashjan, The general solution of two-dimensional matrix Toda chain equations with fixed ends, *Lett. Math. Phys.* **35** (1995) 345–349.

[8] P. Etingof, I. Gelfand and V. Retakh, Factorization of differential operators, quasideterminants, and nonabelian Toda field equations, *Math. Res. Lett.* **4** (1997) 413–425; q-alg/9701008.

[9] P. Etingof, I. Gelfand and V. Retakh, Nonabelian integrable systems, quasideterminants and Marchenko lemma, *Math. Res. Lett.* **5** (1998) 1–12; q-alg/9707017.

[10] J.–L. Gervais and M. V. Saveliev, Higher grading generalizations of the Toda systems, *Nucl. Phys.* **B 453** (1995) 449–476; hep-th/9505047.

[11] A. N. Leznov, Two-dimensional Ultra-Toda integrable mappings and chains (Abelian case), hep-th/9703025.

[12] J. M. Evans and J. O. Madsen. On the classification of real forms of non-Abelian Toda theories and W-algebras, *Nucl. Phys.* **B 536** (1999) 657–703; hep-th/9802201.